A MONOTONICITY APPROACH TO NONLINEAR DIRICHLET PROBLEMS IN PERFORATED DOMAINS

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Abstract. We study the asymptotic behaviour of solutions to Dirichlet problems in perforated domains for nonlinear elliptic equations associated with monotone operators. The main difference with respect to the previous papers on this subject is that no uniformity is assumed in the monotonicity condition. Under a very general hypothesis on the holes of the domains, we construct a limit equation, which is satisfied by the weak limits of the solutions. The additional term in the limit problem depends only on the local behaviour of the holes, which can be expressed in terms of suitable nonlinear capacities associated with the monotone operator.

Introduction

This paper continues previous investigations of the authors on nonlinear Dirichlet problems in perforated domains of general structure.

Let Ω be any bounded domain in the n-dimensional Euclidean space R^n and let $\Omega_s \subset \Omega$, $s = 1, 2, ...$ be a sequence of subdomains. In Ω_s we consider a nonlinear elliptic boundary value problems for $s = 1, 2, ...$

$$\sum_{j=1}^{n} \frac{d}{dx_j} a_j\left(x, \frac{\partial u}{\partial x}\right) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j(x), \;\; x \in \Omega_s, \tag{0.1}$$

$$u(x) = f(x), \;\; x \in \partial\Omega_s. \tag{0.2}$$

Our conditions on the data of problems (0.1), (0.2) provide the existence of a solution $u_s(x) \in W_m^1(\Omega_s)$ for every s and also the boundedness of the sequence $u_s(x)$ in $W_m^1(\Omega)$.

In the previous works on this subject [1-4, 6-10] (see also the References in [6]) the homogenization problems for nonlinear elliptic second order equations were studied under strong monotonicity assumption for the equations. The following inequality

$$\sum_{j=1}^{n} \left[a_j(x,p) - a_j(x,q)\right] (p_j - q_j) \geqslant \nu\, |p-q|^m \tag{0.3}$$

was assumed for arbitrary $p, q \in R^n,\ x \in \Omega$, with a positive constant ν.

In particular, inequality (0.3) guarantees in [6-10] the strong convergence to zero in $W_m^1(\Omega)$ of the remainder term of the asymptotic expansion and the strong convergence of the gradients of solutions of the problems (0.1), (0.2) in $W_p^1(\Omega)$ for $p < m$.

In this paper we assume only the following weak monotonicity condition: for arbitrary points $x \in \Omega,\ p, q \in R^n$ the inequality

$$\sum_{j=1}^{n} \left[a_j(x,p) - a_j(x,q)\right] (p_j - q_j) \geqslant 0 \tag{0.4}$$

is satisfied. This weak condition does not allow us to apply the methods from [6-10], which are based on the study of the behaviour of the asymptotic expansion of the solutions. We develop a new approach by using monotonicity arguments. This allows us to construct a boundary value problem (in a fixed domain), which is satisfied by the weak limits of subsequences of $u_s(x)$.

This approach is based on the construction of special test functions and on the analysis of their behaviour. For this analysis we use precise pointwise and integral estimates for the potential functions which are solutions of some auxiliary boundary value problems in domains with holes of small diameter.

We call the attention of the reader to the main result of this paper, the Convergence Theorem (Theorem 1.1), that is proved by using a new pointwise estimate (Lemma 2.3) of the potential functions. This theorem allows us to make the main modification in the construction of the corrector, if assumption (0.3) is satisfied. Note that in the previous papers [6-10] the definition of the subdivision of the domain, and consequently the construction of the corrector, depended on sequence $u_s(x)$. The subdivision and the corrector we construct in the present paper, by using the Convergence Theorem, are independent of $u_s(x)$.

Our assumption on the perforated domains (see Condition B in Section 1) coincides with the corresponding condition in [10]. We suppose that the C_m-capacity of the

portion of the holes in any small cube is estimated from above by the Lebesgue measure of the cube.

We construct the limit boundary value problem, and we describe the additional term which appears in it by means of some quantitative capacitary properties of the holes.

1. Statement of the results

We assume that the functions $a_j(x,p)$, $j = 1, ..., n$, are defined for $x \in R^n$, $p \in R^n$, and satisfy the following conditions:

Condition A.1. *The functions $a_j(x,p)$ are continuous in p for all $x \in R^n$ and measurable in x for all $p \in R^n$.*

Condition A.2. *There exist two positive constants ν_1, ν_2, and a constant m, with $2 \leqslant m < n$, such that*

$$\sum_{j=1}^{n} a_j\,(x,p)\,p_j \geqslant \nu_1\,(1 + |p|)^{m-2} \cdot |p|^2, \tag{1.1}$$

$$\sum_{j=1}^{n} \big[a_j\,(x,p) - a_j\,(x,q)\big]\,(p_j - q_j) \geqslant 0, \tag{1.2}$$

$$\sum_{j=1}^{n} \big|a_j\,(x,p) - a_j\,(x,q)\big| \leqslant \nu_2\,(1 + |p| + |q|)^{m-2} \cdot |p - q|, \tag{1.3}$$

for every $x \in R^n$, $p, q \in R^n$.

Note that from (1.1) it follows that $a_j\,(x,0) = 0$ for every $x \in R^n$. Therefore (1.3) implies that

$$|a_j\,(x,p)| \leqslant \nu_2\,(1 + |p|)^{m-2}\,|p| \tag{1.4}$$

for every $x \in R^n$, $p \in R^n$, $j = 1, ..., n$.

We assume that functions $f_j(x)$, $j = 1, ..., n$, and $f(x)$ in (0.1), (0.2) are defined in R^n and satisfy the conditions:

$$f_j(x) \in L_{m'}(R^n), \qquad f(x) \in W_m^1(R^n) \tag{1.5}$$

for $j = 1, ..., n$ and $m' = \frac{m}{m-1}$.

A solution of the boundary value problem (0.1), (0.2) is a function $u(x) \in W_m^1(\Omega_s)$, satisfying $u(x) - f(x) \in \overset{\circ}{W}{}_m^1(\Omega_s)$, such that the integral identity

$$\sum_{j=1}^{n} \int_{\Omega_s} \Big[a_j \big(x, \frac{\partial u}{\partial x}\big) - f_j(x) \Big] \frac{\partial \varphi(x)}{\partial x_j} \, dx = 0 \tag{1.6}$$

holds for an arbitrary function $\varphi(x) \in \overset{\circ}{W}{}_m^1(\Omega_s)$.

Using methods of the theory of monotone operators it is easy to prove the existence of a solution of problem (0.1), (0.2). For every s we denote by $u_s(x)$ one of the possible solution of the problem (0.1), (0.2) and extend $u_s(x)$ on R^n by setting $u_s(x) = f(x)$ for $x \in R^n \setminus \Omega_s$. By condition A.2 and (1.5) the estimate

$$\int_{R^n} \Big\{ \Big| \frac{\partial u_s(x)}{\partial x} \Big|^m + |u_s(x)|^m \Big\} \, dx \leqslant R \tag{1.7}$$

holds with a constant R independent of s.

By (1.7) the sequence $u_s(x)$ contains a weakly convergent subsequence, therefore we may assume that $u_s(x)$ converges weakly in $W_m^1(R^n)$ to some function $u_0(x)$.

We formulate now our assumptions on the sequence Ω_s in terms of the m-capacity $C_m(F)$. For every compact set F, its m-capacity $C_m(F)$ is defined by

$$C_m(F) = \inf \int_{R^n} \Big| \frac{\partial \varphi(x)}{\partial x} \Big|^m \, dx \,, \tag{1.8}$$

where the infimum is taken over all function $\varphi(x) \in C_0^\infty(R^n)$ which satisfy the condition $\varphi(x) = 1$ for $x \in F$.

For every $x_0 = (x_1^{(0)}, ..., x_n^{(0)}) \in R^n$, $r > 0$, we set

$$K(x_0, r) = \{x \in R^n : |x_j - x_j^{(0)}| \leqslant r, \; j = 1, ..., n\}. \tag{1.9}$$

Let us assume that the following condition is satisfied.

Condition B. *There exist a positive number A and a sequence $r_s > 0$, tending to zero as $s \to \infty$, such that the inequality*

$$C_m(K(x, r) \setminus \Omega_s) \leqslant A \, r^n \tag{1.10}$$

holds for every $x \in \Omega$ and for every $r \geqslant r_s$ with $K(x, r + r_s) \subset \Omega$.

Let us fix a bounded open set $\Omega_0 \subset R^n$ such that $\rho(\partial\Omega_0, \Omega) \geqslant 1$, where $\rho(\partial\Omega_0, \Omega)$ is the distance from $\partial\Omega_0$ to Ω, and let $\psi(x)$ be a function of class $C_0^\infty(\Omega_0)$ equal to 1 on $\overline{\Omega}$. For every compact set F contained in Ω and for every real number q we define the auxiliary function $v(x, F, q)$ as a solution of the boundary value problem

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\, a_j\big(x, \frac{\partial v}{\partial x}\big) = 0, \quad x \in \Omega_0 \setminus F\,, \tag{1.11}$$

$$v(x) = q\,\psi(x), \quad x \in \partial(\Omega_0 \setminus F)\,. \tag{1.12}$$

The solvability of problem (1.11), (1.12) follows easily from the theory of monotone operators. In [5] it is proved that this problem admits a maximal solution, i.e., there exists a solution $\overline{v}(x)$ of problem (1.11), (1.12) such that $v(x) \leqslant \overline{v}(x)$ for any solution $v(x)$ of the same problem. We denote this maximal solution by $v(x, F, q)$, and extend it to R^n by setting $v(x, F, q) = q$ in F and $v(x, F, q) = 0$ outside Ω_0.

In Section 3 we shall introduce a special decomposition of the domain Ω of the form

$$\Omega = \big\{ \bigcup_{\alpha \in I_s} K(x_\alpha^{(s)}, \lambda_s \rho_s) \big\} \cup U_s \tag{1.13}$$

where λ_s and ρ_s are sequences of positive real numbers such that $\lambda_s \to \infty$, $\rho_s \to 0$ and $\lambda_s \rho_s \to 0$ as $s \to \infty$, $x_\alpha^{(s)} = 2\lambda_s \rho_s \alpha$, $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index with integer coordinates, I_s is the set of all multi-indices α such that $K(x_\alpha^{(s)}, 2\lambda_s\rho_s) \subset \Omega$, and U_s is the complement of $\bigcup_{\alpha \in I_s} K(x_\alpha^{(s)}, 2\lambda_s\rho_s)$ with respect to Ω.

We define $v_\alpha^{(s)}(x, q) = v(x, F, q)$ for $F = K(x_\alpha^{(s)}, (\lambda_s - 2)\rho_s) \setminus \Omega_s$. Let $q_s(x)$ be an arbitrary sequence that converges strongly in $L_m(\Omega)$ and let $q_\alpha^{(s)}$ be the mean value of the function $q_s(x)$ in the cube $K(x_\alpha^{(s)}, \lambda_s\rho_s)$.

In Section 3 we shall construct the following sequence, which is fundamental in our analysis:

$$r_s(x) = \sum_{\alpha \in I_s} v_\alpha^{(s)}(x, q_\alpha^{(s)})\, \varphi_\alpha^{(s)}(x)\,, \tag{1.14}$$

where $\varphi_\alpha^{(s)}(x)$ is a special cut-off function, constructed by using $v_\alpha^{(s)}(x, q_\alpha^{(s)})$ (see (3.6)), which is equal to 1 for $x \in K(x_\alpha^{(s)}, (\lambda_s - 2)\rho_s) \setminus \Omega_s$ and equal to 0 outsite $K(x_\alpha^{(s)}, \lambda_s\rho_s)$. Remark that $r_s(x)$ is analogous with the corrector which was constructed in [6,9]. In Section 3 we shall prove the following result.

Theorem 1.1 (Convergence Theorem). *Assume that conditions A.1, A.2, and B are satisfied and let $q_s(x)$ be some sequence converging strongly in $L_m(\Omega)$. Let $z_s(x)$ be an arbitrary sequence of functions such that $z_s(x) \in \overset{\circ}{W}{}^1_m(\Omega_s)$ and $z_s(x)$ converges weakly to zero in $W^1_m(\Omega)$. Then*

$$\lim_{s\to\infty} \sum_{j=1}^{n} \int\limits_{\Omega} a_j\big(x, \frac{\partial r_s(x)}{\partial x}\big)\ \frac{\partial z_s(x)}{\partial x_j}\ dx = 0. \tag{1.15}$$

In order to formulate a result about the boundary value problem for the function $u_0(x)$ we introduce a capacity connected with the differential equation (0.1), defined for every compact set $F \subset \Omega$ and for every real number $q \neq 0$ by the equality

$$C_A(F,q) = \sum_{j=1}^{n} \frac{1}{q} \int\limits_{\Omega} a_j\big(x, \frac{\partial v(x,F,q)}{\partial x}\big)\ \frac{\partial}{\partial x_j}\ v(x,F,q)\ dx\,, \tag{1.16}$$

where $v(x,F,q)$ is the maximal solution of the problem (1.11), (1.12), $C_A(F,0) = 0$. For the main properties of this capacity, in particular the continuity with respect to q, we refer to [5].

We assume that the following condition is satisfied.

Condition C. *There exists a function $c(x,q)$, continuous in $x, q \in \Omega \times R^1$, such that for an arbitrary point $x \in \Omega$ and an arbitrary $q \in R^1$ we have*

$$\lim_{r\to 0} \Big\{ \lim_{s\to\infty} \frac{1}{\operatorname{meas} K(x,r)} C_A\big(K(x,r) \setminus \Omega_s, q\big) \Big\} = c(x,q)\,, \tag{1.17}$$

and the convergences to the limits in (1.17) are uniform with respect to q on any bounded interval and with respect to $x \in \Omega$.

The main result of the paper, proved in Section 5, is the following theorem.

Theorem 1.2. *Assume that conditions A.1, A.2, B, C and (1.5) are satisfied. Let $u_s(x)$ be a sequence of solutions of the problem (0.1), (0.2) which converges weakly in $W^1_m(\Omega)$ to a function $u_0(x)$. Then the function $u_0(x)$ is a solution of the problem*

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j}\ a_j\big(x, \frac{\partial u}{\partial x}\big) + c(x, f(x) - u(x)) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\ f_j(x),\ \ x \in \Omega, \tag{1.18}$$

$$u(x) = f(x),\ \ x \in \partial\Omega, \tag{1.19}$$

where $c(x,q)$ is the function defined by (1.17).

Remark 1.3. It is possible to establish all results of this paper if one replaces inequalities (1.1), (1.3) by the inequalities

$$\sum_{j=1}^{n} a_j(x,p)p_j \geqslant \nu_1|p|^m, \tag{1.20}$$

$$\sum_{j=1}^{n} |a_j(x,p) - a_j(x,q)| \leqslant \nu_2(|p|+|q|)^{m-2} \cdot |p-q|. \tag{1.21}$$

Remark 1.4. If we assume that conditions A.1 and B hold, that inequalities (1.2), (1.20), (1.21) are satisfied for all $x \in \Omega$, $p,q \in R^n$, and that $a_j(x,p)$ are odd and $(m-1)$-homogeneous with respect to p, then the capacity defined by (1.16) satisfies the following equality

$$C_A(F,\lambda q) = |\lambda|^{m-2}\lambda C_A(F,q) \tag{1.22}$$

for every $q, \lambda \in R^1$.

Under these assumptions we can formulate condition C in the following weak form.

Condition C′. *There exists a measurable function $c(x)$ such that for almost every $x \in \Omega$*

$$\begin{aligned} &\lim_{r\to 0}\left\{\liminf_{s\to\infty} \frac{1}{\operatorname{meas} K(x,r)} C_A\big(K(x,r)\setminus\Omega_s, 1\big)\right\} = \\ =&\lim_{r\to 0}\left\{\lim\mathit{sup}_{s\to\infty} \frac{1}{\operatorname{meas} K(x,r)} C_A\big(K(x,r)\setminus\Omega_s, 1\big)\right\} = c(x). \end{aligned} \tag{1.23}$$

If all assumptions of this Remark are satisfied, it is still possible to prove the result of the Theorem 1.2. For the changes in the proof we refer to the discussions of Section 6 in [6].

2. Estimates for potentials and averaging functions

In this section we establish some integral and pointwise estimates for the potential functions $v(x,F,q)$ introduced in Section 1 as solutions of problems (1.11), (1.12).

Throughout the paper we shall use the notation C_j, $j = 1, 2, \dots$, to indicate a constant which depends only on n, m, ν_1, ν_2, A, R, meas Ω (see (1.1), (1.3), (1.7), (1.10)).

Let us fix a compact set F contained in Ω and let $v(x,q) = v(x,F,q)$. For $\mu > 0$ we define the set

$$E(\mu) = \{x \in \Omega_0 : |v(x,q)| \leqslant \mu\}. \tag{2.1}$$

Lemma 2.1. *Assume that conditions A.1, A.2 are satisfied and that diam $(F) \leqslant r$. Then there exists a constant K_1, depending only on ν_1, ν_2, n, m, such that the estimate*

$$\int\limits_{E(\mu)} \left(1 + \left|\frac{\partial v(x,q)}{\partial x}\right|\right)^{m-2} \cdot \left|\frac{\partial v(x,q)}{\partial x}\right|^2 dx \leqslant \; K_1 \mu |q| (|q| + r)^{m-2} \, C_m(F) \tag{2.2}$$

holds for every $q \in R^1$ and for every $\mu > 0$.

Proof. See [6], Lemma 2.1.

It is easy to see that the inequality $0 \leqslant \frac{1}{q} v(x,q) \leqslant 1$ holds for every $q \neq 0$ and a.e. $x \in \Omega_0$. So we obtain an estimate of the norm of the function $v(x,q)$ in $W^1_m(\Omega_0)$ if we put $\mu = |q|$ in (2.2).

Theorem 2.2. *Assume that conditions A.1, A.2 are satisfied, and that F is contained in a cube $K(x_0, r)$. Then there exists a constant K_2, depending only on ν_1, ν_2, n, m, such that for every $x \in K(x_0, 3r) \setminus K(x_0, r)$ we have*

$$|v(x,q)| \leqslant K_2 |q| \cdot \left[\frac{r}{\rho(x, K(x_0, r))}\right]^{n-1} \cdot \left[\frac{C_m(F)}{r^{n-m}}\right]^{\frac{1}{m-1}}, \tag{2.3}$$

where $\rho(x, K(x_0, r))$ is the distance from the point x to the cube $K(x_0, r)$.

Proof. See [10], Theorem 2.5.

Lemma 2.3. *Assume that the conditions of Theorem 2.2 and the inequalities*

$$C_m(F) \leqslant A\, r^n, \quad |q|^{m-1} r \leqslant 1 \tag{2.4}$$

are satisfied. Then there exists a constant K_3, depending only on ν_1, ν_2, n, m and A, such that the estimate

$$|v(x,q)| \leqslant K_3 |q| [|q| + r]^{m-2} \cdot r^2 \tag{2.5}$$

holds for $x \in K(x_0, 2r) \setminus K(x_0, \frac{3r}{2})$.

Proof. We consider the case $q>0$. For $\frac{r}{2}<\rho<r$ we define two numerical sequences

$$\rho_j^{(1)}=\frac{\rho}{2}[1+2^{-j}],\ \rho_j^{(2)}=\frac{\rho}{2}[3-2^{-j}],\ j=1,2,\dots,$$

and smooth functions $\varphi_j(x)$, equal to one on the set $G_j=K(x_0,r+\rho_j^{(2)})\setminus K(x_0,r+\rho_j^{(1)})$, vanishing outside G_{j+1}, and such that $0\leqslant\varphi_j(x)\leqslant 1$, $\left|\frac{\partial\varphi_j(x)}{\partial x}\right|\leqslant\frac{2}{\rho}^{j+3}$.

Let us use the test function $[v(x,q)]^{\sigma+1}[\varphi_j(x)]^{\tau+m}$ in the integral identity corresponding to the boundary value problem (1.11), (1.12), where σ,τ are arbitrary numbers greater than one. Estimating by means of condition A.2 and Young's inequality we obtain

$$\begin{aligned}(2.6)\qquad &\int\limits_{G_{j+1}}\big[1+\big|\frac{\partial v}{\partial x}\big|\big]^{m-2}\,\big|\frac{\partial v}{\partial x}\big|^2 v^\sigma\,\varphi_j^{\tau+m}\,dx\leqslant\\ &\leqslant C_1\tau^m\int\limits_{G_{j+1}}\Big[v^{\sigma+2}\big(\frac{2^j}{\rho}\big)^2\varphi_j^{\tau+m-2}+v^{\sigma+m}\big(\frac{2^j}{\rho}\big)^m\varphi_j^\tau\Big]\,dx\,.\end{aligned}$$

We can estimate $v(x)$ on the set G_{j+1} by using inequalities (2.3), (2.4) and we obtain $v(x,q)\leqslant C_2\rho$, which, together with (2.6), yields

$$\int\limits_{G_{j+1}}\big|\frac{\partial v}{\partial x}\big|^2 v^\sigma\,\varphi_j^{\tau+m}\,dx\leqslant C_3\tau^m\frac{2^{jm}}{r^2}\int\limits_{G_{j+1}}v^{\sigma+2}\cdot\varphi_j^\tau\,dx\,. \tag{2.7}$$

Define

$$m_j=\text{ess sup}\{v(x,q):x\in G_j\}\,. \tag{2.8}$$

From inequality (2.7) and Lemma 2.7 of [10] we obtain the following estimate

$$m_j^2\leqslant C_4\frac{2^{\frac{jmn}{2}}}{r^n}\int\limits_{G_{j+1}}v^2\varphi_j^2dx. \tag{2.9}$$

The integral in the right-hand side of the last inequality is estimated using Poincaré's inequality (see, e.g., [8], Chapter 8, Lemma 1.4) and (2.2):

$$\begin{aligned}(2.10)\qquad &\int\limits_{G_{j+1}}v^2\varphi_j^2dx\leqslant\int\limits_{G_{j+1}}|\min(v(x,q),m_{j+1})|^2dx\leqslant\\ &\leqslant C_5r^2\int\limits_{E(m_{j+1})}\left|\frac{\partial v(x,q)}{\partial x}\right|^2dx\,\leqslant C_6m_{j+1}q[q+r]^{m-2}\,\cdot\,r^{2+n}.\end{aligned}$$

By virtue of inequalities (2.9), (2.10) we have the estimate

$$m_j^2 \leqslant C_7\, 2^{\frac{jmn}{2}}\, m_{j+1} q[q+r]^{m-2} \cdot r^2 \text{ for } j = 1, 2, \dots, \tag{2.11}$$

whereby, using Lemma 2.9 of [10], it follows that

$$m_1 \leqslant C_8\, q\, [q+r]^{m-2} \cdot r^2. \tag{2.12}$$

In conclusion, we obtain estimate (2.5) from (2.8), (2.12) and the definition of G_1. This completes the proof of lemma.

We shall now state some properties of the averaging function $u_h(x)$ defined by

$$u_h(x) = \frac{1}{h^n} \int\limits_{R^n} K\left(\frac{|x-y|}{h}\right) u(y)\, dy\,, \tag{2.13}$$

where $K(t)$ is an infinitely differentiable function, equal to zero for $|t| \geqslant 1$, such that

$$\int\limits_{R^n} K(|x|)\, dx = 1$$

and $0 \leqslant K(t) \leqslant c(n)$ for a suitable constant $c(n)$ depending only on n.

For a given positive number h, let us consider the family of points $x_\alpha = 2h\alpha$ in R^n, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with integer coordinates. Let $I(h)$ be the set of multi-indices α such that $K(x_\alpha, 2h) \subset \Omega$ and, for every integrable function $u(x)$, let

$$u(\alpha, h) = \frac{1}{[2h]^n} \int\limits_{K(x_\alpha, h)} u_h(x)\, dx$$

be the mean value of $u_h(x)$ in the cube $K(x_\alpha, h)$, where $u_h(x)$ is defined by (2.13).

Lemma 2.4. *Let θ be a constant with $1 \leqslant \theta \leqslant 2$ and let $u(x), g(x)$ be functions from the spaces $W_m^1(\Omega), L_m(\Omega)$ respectively. Assume that, for some positive constant Q, the inequalities*

$$\int\limits_{K(x_\alpha, \theta h)} |g(x)|^m\, dx \leqslant Q h^n,\ \alpha \in I(h) \tag{2.14}$$

are satisfied. Then there exists a constant K_4, depending only on n, m, such that

$$\begin{aligned} \sum_{\alpha \in I(h)} \int\limits_{K(x_\alpha, \theta h)} |u_h(x) - u(\alpha, \theta h)|^m \cdot |g(x)|^m\, dx \leqslant \\ \leqslant K_4 Q \cdot h^m \int\limits_\Omega \Big|\frac{\partial u(x)}{\partial x}\Big|^m\, dx\,. \end{aligned} \tag{2.15}$$

Proof. See [6], Lemma 2.7.

3. Proof of the Convergence Theorem

Let us define the sequences ρ_s, μ_s, λ_s, $s = 1, 2, \dots$, by

$$\lim_{s\to\infty} \rho_s = 0, \qquad \rho_s \geqslant r_s, \qquad \mu_s = \Big[\ln \frac{1}{\rho_s}\Big]^{-1}, \qquad \lambda_s = \Big\{E\Big(\ln \frac{1}{\rho_s}\Big)\Big\}^{2m}, \tag{3.1}$$

where r_s is the number which appears in the condition B and $E\big(\ln \frac{1}{\rho_s}\big)$ denotes the integer part of the number $\ln \frac{1}{\rho_s}$.

We consider the subdivision of the domain Ω introduced in (1.13) and we denote

$$K_s(\alpha) = K(x_\alpha^{(s)}, \lambda_s \rho_s), \qquad K_s'(\alpha) = K(x_\alpha^{(s)}, (\lambda_s - 2)\rho_s). \tag{3.2}$$

Let $q_s(x)$ be an arbitrary sequence in $L_m(\Omega)$ that converges strongly in $L_m(\Omega)$ to some function $q_0(x)$. We introduce the sets I_s', I_s'' of multi-indices by

$$I_s' = \{\alpha \in I_s : |q_\alpha^{(s)}| > 2\mu_s\}, \ I_s'' = \{\alpha \in I_s : |q_\alpha^{(s)}| \leqslant 2\mu_s\}\,, \tag{3.3}$$

where $q_\alpha^{(s)}$ is the mean value of the function $q_s(x)$ in the cube $K_s(\alpha)$. Let us define the functions $w_\alpha^{(s)}(x)$, $\alpha \in I_s$, by

$$w_\alpha^{(s)}(x) = v_\alpha^{(s)}(x, \tilde{q}_\alpha^{(s)}), \tag{3.4}$$

where

$$\tilde{q}_\alpha^{(s)} = q_\alpha^{(s)} \quad \text{for} \quad \alpha \in I_s', \quad \tilde{q}_\alpha^{(s)} = 2\mu_s \quad \text{for} \quad \alpha \in I_s''. \tag{3.5}$$

For an arbitrary function $g(x)$ we denote its positive part by $[g(x)]_+ = \max\{g(x), 0\}$. We define the cut-off functions $\varphi_\alpha^{(s)}(x)$ by

$$\varphi_\alpha^{(s)}(x) = \frac{2}{\mu_\alpha^{(s)}} \min\left\{\left[|w_\alpha^{(s)}(x)| - \frac{\mu_\alpha^{(s)}}{2}\right]_+, \ \frac{\mu_\alpha^{(s)}}{2}\right\}, \tag{3.6}$$

where

$$\mu_\alpha^{(s)} = \mu_s \cdot \max\{1, |q_\alpha^{(s)}|\}. \tag{3.7}$$

Let $G_\alpha^{(s)}$ be the support of the function $\varphi_\alpha^{(s)}(x)$.

Lemma 3.1. *Assume that conditions A.1, A.2, and B are satisfied. Then there exists a number s_1 such that the inclusions*

$$G_\alpha^{(s)} \subset K(x_\alpha^{(s)}, \ (\lambda_s - 1)\rho_s) \quad \textit{for} \quad \alpha \in I_s \tag{3.8}$$

hold for $s \geqslant s_1$.

The proof is analogous with the proof of Lemma 4.1 in [10].

Lemma 3.2. *Assume that conditions A.1, A.2, and B are satisfied. Then the inequalities*

$$\operatorname{meas} G_\alpha^{(s)} \leqslant K_5[\lambda_s\rho_s]^{m+n}\cdot\mu_s^{1-m} \quad \textit{for} \quad \alpha\in I_s \tag{3.9}$$

hold with a constant K_5 *depending only on* ν_1,ν_2,n,m,A.

The proof is analogous with the proof of Lemma 4.2 in [10].

Lemma 3.3. *Assume that conditions A.1, A.2, and B are satisfied, and let* $q_s(x)$ *be an arbitrary sequence converging strongly in* $L_m(\Omega)$ *as* $s\to\infty$. *Then the sequence* $r_s(x)$ *defined by (1.14) converges to zero weakly in* $W_m^1(\Omega)$ *and strongly in* $W_p^1(\Omega)$ *for any* $p<m$.

Proof. We can assume that $s\geqslant s_1$, where s_1 is defined in Lemma 3.1. Then from inclusions (3.8) we have

$$G_\alpha^{(s)}\cap G_\beta^{(s)}=\varnothing \quad \text{for} \quad \alpha\neq\beta,\ \alpha,\beta\in I_s. \tag{3.10}$$

Let us estimate the norm of the gradient of $r_s(x)$ in $L_m(\Omega)$ for s large enough such that

$$1\geqslant 2\mu_s>\mu_s\geqslant\lambda_s\rho_s. \tag{3.11}$$

We have

$$\begin{aligned}\left\|\frac{\partial r_s(x)}{\partial x}\right\|_{L_m(\Omega)}^m &\leqslant C_9\cdot\sum_{\alpha\in I_s}\int\limits_{G_\alpha^{(s)}}\left|\frac{\partial v_\alpha^{(s)}(x,q_\alpha^{(s)})}{\partial x}\right|^m dx\,+\\ &+C_9\cdot\sum_{\alpha\in I_s}\left[\mu_\alpha^{(s)}\right]^{-m}\int\limits_{\tilde E_\alpha^{(s)}}|v_\alpha^{(s)}(x,q_\alpha^{(s)})|^m\cdot\left|\frac{\partial v_\alpha^{(s)}(x,\tilde q_\alpha^{(s)})}{\partial x}\right|^m dx\,,\end{aligned} \tag{3.12}$$

where $\tilde E_\alpha^{(s)}=\left\{x\in\Omega_0:\mu_\alpha^{(s)}/2\leqslant|v_\alpha^{(s)}(x,\tilde q_\alpha^{(s)})|\leqslant\mu_\alpha^{(s)}\right\}$.

The first term in the right-hand side of (3.12) is estimated by using inequality (2.2) and condition B:

$$\sum_{\alpha\in I_s}\int\limits_{G_\alpha^{(s)}}\left|\frac{\partial v_\alpha^{(s)}(x,q_\alpha^{(s)})}{\partial x}\right|^m dx\ \leqslant\ C_{10}\sum_{\alpha\in I_s}\left(|q_\alpha^{(s)}|^m+1\right)[\lambda_s\rho_s]^n. \tag{3.13}$$

From Hölder's inequality we have

$$|q_\alpha^{(s)}| = \frac{1}{[2\lambda_s\rho_s]^n}\left|\int\limits_{K_s(\alpha)} q_s(x)dx\right| \leqslant \frac{1}{[2\lambda_s\rho_s]^{\frac{n}{m}}}\left\{\int\limits_{K_s(\alpha)} |q_s(x)|^m\,dx\right\}^{\frac{1}{m}} \tag{3.14}$$

and we estimate the sum in the right-hand side of (3.13) by

$$\sum_{\alpha\in I_s} |q_\alpha^{(s)}|^m\ [2\lambda_s\rho_s]^n \ \leqslant\ \int\limits_\Omega |q_s(x)|^m\,dx, \qquad \sum_{\alpha\in I_s} [2\lambda_s\rho_s]^n \ \leqslant\ \operatorname{meas}\Omega\,.$$

Recalling the inequality

$$|v_\alpha^{(s)}(x,q_\alpha^{(s)})| \ \leqslant\ \mu_\alpha^{(s)} \quad\text{for}\quad x\in\tilde{E}_\alpha^{(s)},\ \alpha\in I_s\,, \tag{3.15}$$

we can estimate the second sum in the right-hand side of (3.12) as in (3.13) and we obtain

$$\int\limits_\Omega \left|\frac{\partial r_s(x)}{\partial x}\right|^m dx \ \leqslant\ C_{11}\int\limits_\Omega \big(|q_s(x)|^m+1\big)\,dx. \tag{3.16}$$

Since the function $r_s(x)$ vanishes outside $\underset{\alpha\in I_s}{\cup} G_\alpha^{(s)}$, applying Hölder's inequality we deduce that, for $1<p<m$,

$$\|\frac{\partial r_s(x)}{\partial x}\|_{L_p(\Omega)} \ \leqslant\ \|\frac{\partial r_s(x)}{\partial x}\|_{L_m(\Omega)}\ \cdot\ \{\sum_{\alpha\in I_s}\operatorname{meas} G_\alpha^{(s)}\}^{\frac{1}{p}-\frac{1}{m}}.$$

The right-hand side of this inequality tends to zero by (3.1), (3.9), and (3.16).

Since, by (3.8), $r_s(x)$ has compact support in Ω for $s\geqslant s_1$, the conclusions of the lemma follow from Poincaré's inequality and Rellich's compactness theorem.

Let ζ_s be an arbitrary sequence in R^1 such that

$$\lim_{s\to\infty}\zeta_s = 0. \tag{3.17}$$

Let us define the sets $I'_{1,s}$, $I'_{2,s}$ of multi-indices by

$$I'_{1,s} = \big\{\alpha\in I'_s : \zeta_s|q_\alpha^{(s)}|^{m-1}\ \leqslant 1\big\},\quad I'_{2,s} = \big\{\alpha\in I'_s : \zeta_s|q_\alpha^{(s)}|^{m-1}\ > 1\big\}, \tag{3.18}$$

and denote

$$r'_{i,s}(x)\ =\ \sum_{\alpha\in I'_{i,s}} v_\alpha^{(s)}(x,q_\alpha^{(s)})\ \varphi_\alpha^{(s)}(x),\quad i=1,2. \tag{3.19}$$

Lemma 3.4. *Assume that the conditions of Lemma 3.3 are satisfied and let ζ_s be an arbitrary sequence in R^1 satisfying (3.17). Then the sequence $r'_{2,s}(x)$ defined by (3.19) converges strongly to zero in $W^1_m(\Omega)$.*

Proof. Define

$$Q_s = \bigcup_{\alpha \in I'_{2,s}} K_s(\alpha)\,. \tag{3.20}$$

From (3.14) and from $\zeta_s^{-\frac{m}{m-1}} \operatorname{meas} Q_s \leqslant C_{12} \sum_{\alpha \in I'_{2,s}} |q^{(s)}_\alpha|^m \, [\lambda_s \rho_s]^n$ we get

$$\operatorname{meas} Q_s \leqslant C_{12}\, \zeta_s^{\frac{m}{m-1}} \int_\Omega |q_s(x)|^m \, dx. \tag{3.21}$$

As in the proof of inequality (3.16), we obtain

$$\int_\Omega \left| \frac{\partial r'_{2,s}(x)}{\partial x} \right|^m dx \leqslant C_{13} \int_{Q_s} \big(|q_s(x)|^m + 1 \big)\, dx\,,$$

and the convergence to zero of the right-hand side of the last inequality follows from (3.17), (3.21), and the assumption on the sequence $q_s(x)$. The proof of the lemma is complete.

Lemma 3.5. *Assume that the conditions of Lemma 3.3 are satisfied. Then the sequence*

$$r''_s(x) = \sum_{\alpha \in I''_s} v^{(s)}_\alpha(x, q^{(s)}_\alpha)\, \varphi^{(s)}_\alpha(x) \tag{3.22}$$

converges strongly to zero in $W^1_m(\Omega)$.

The proof follows immediately from the estimate

$$\int_\Omega \left| \frac{\partial r''_s(x)}{\partial x} \right|^m dx \leqslant C_{14} \sum_{\alpha \in I''_s} \big(\mu_s^m + [\lambda_s \rho_s]^m \big) [\lambda_s \rho_s]^n \leqslant C_{14} \big(\mu_s^m + [\lambda_s \rho_s]^m \big) \operatorname{meas} \Omega\,,$$

that is obtained as in (3.13), using the definition of the set I''_s in (3.3).

Proof of Theorem 1.1. Define the sequence ζ_s by

$$\zeta_s = \max\left\{ ||z_s(x)||_{L_m(\Omega)},\ \lambda_s \rho_s \right\}, \tag{3.23}$$

where $z_s(x)$ is the sequence introduced in the statement of Theorem 1.1. Then ζ_s tends to zero as $s \to \infty$. Let $r'_{1,s}(x)$, $r'_{2,s}(x)$ be the sequences defined by (3.19) for this choice of ζ_s.

Using condition A.2, Lemmas 3.3–3.5, and the assumptions on $z_s(x)$ we obtain

$$\lim_{s\to\infty} \sum_{j=1}^{n} \int_{\Omega} \Big[a_j\big(x, \frac{\partial r_s(x)}{\partial x}\big) - a_j\big(x, \frac{\partial r'_{1,s}(x)}{\partial x}\big)\Big] \, \frac{\partial z_s(x)}{\partial x_j} \, dx = 0\,, \tag{3.24}$$

and it is sufficient to study the behaviour of the term

$$J_s = \sum_{j=1}^{n} \int_{\Omega} a_j\big(x, \frac{\partial r'_{1,s}(x)}{\partial x}\big) \, \frac{\partial z_s(x)}{\partial x_j} \, dx. \tag{3.25}$$

Let $\eta_\alpha^{(s)}(x)$ be a function of class $C_0^\infty(\Omega_0)$, which is equal to one on $K\bigg(x_\alpha^{(s)},\, \frac{3\lambda_s\rho_s}{2}\bigg)$, to zero outside $K(x_\alpha^{(s)},\, 2\lambda_s\rho_s)$, and such that $\left|\frac{\partial \eta_\alpha^{(s)}(x)}{\partial x}\right| \leqslant \frac{4}{\lambda_s\rho_s}$. We rewrite J_s in the form

$$J_s = \sum_{j=1}^{3} J_s^{(i)}\,, \tag{3.26}$$

where

$$\begin{aligned}
J_s^{(1)} &= \sum_{\alpha\in I'_{1,s}} \sum_{j=1}^{n} \int_{\tilde K_s(\alpha)} \Big[a_j\big(x, \frac{\partial}{\partial x}(v_\alpha^{(s)}\varphi_\alpha^{(s)})\big) - a_j\big(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\big)\Big] \, \frac{\partial z_s(x)}{\partial x_j} \, dx,\\
J_s^{(2)} &= \sum_{\alpha\in I'_{1,s}} \sum_{j=1}^{n} \int_{\tilde K_s(\alpha)} a_j\big(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\big) \, \frac{\partial}{\partial x_j} \, \big[\eta_\alpha^{(s)}(x) z_s(x)\big] \, dx,\\
J_s^{(3)} &= \sum_{\alpha\in I'_{1,s}} \sum_{j=1}^{n} \int_{\tilde K_s(\alpha)} a_j\big(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\big) \, \frac{\partial}{\partial x_j} \big[(1-\eta_\alpha^{(s)}(x)) z_s(x)\big] \, dx;
\end{aligned} \tag{3.27}$$

here $v_\alpha^{(s)} = v_\alpha^{(s)}(x, q_\alpha^{(s)})$ and $\tilde K_s(\alpha) = K(x_\alpha^{(s)}, 2\lambda_s\rho_s)$.

Define $E_\alpha^{(s)}(\mu) = \big\{x \in \tilde K_s(\alpha) : |v_\alpha^{(s)}(x, q_\alpha^{(s)})| \leqslant \mu\big\}$. The function $\varphi_\alpha^{(s)}(x)$ is equal to one if $|v_\alpha^{(s)}(x, q_\alpha^{(s)})| \geqslant \mu_\alpha^{(s)}$, $\alpha \in I'_s$, and using (1.3) and Hölder's inequality we obtain the estimate

$$\begin{aligned}
|J_s^{(1)}| \leqslant C_{15} &\Big\{ \sum_{\alpha\in I'_{1,s}} \int_{E_\alpha^{(s)}(\mu_\alpha^{(s)})} \Big[1 + \big|\frac{\partial}{\partial x}(v_\alpha^{(s)}\varphi_\alpha^{(s)})\big| + \big|\frac{\partial v_\alpha^{(s)}}{\partial x}\big|\Big]^m \, dx \Big\}^{\frac{m-2}{m}} \cdot\\
&\cdot \Big\{ \sum_{\alpha\in I'_{1,s}} \int_{E_\alpha^{(s)}(\mu_\alpha^{(s)})} \big|\frac{\partial}{\partial x}\big[v_\alpha^{(s)}(1-\varphi_\alpha^{(s)})\big]\big|^m \, dx \Big\}^{\frac{1}{m}} \cdot \Big\{ \int_{\Omega} \big|\frac{\partial z_s(x)}{\partial x}\big|^m \, dx \Big\}^{\frac{1}{m}}.
\end{aligned} \tag{3.28}$$

The first factor in the right-hand side of the last inequality can be estimated from above by a constant independent of s. This can be obtained as in the proof of inequality (3.16).

We assume now that s is large enough so that inequality (3.11) is satisfied. The second factor in the right-hand side of (3.28) is estimated using inequalities (2.2), (3.14), and condition B. We obtain

$$\sum_{\alpha\in I'_{1,s}} \int\limits_{E_\alpha^{(s)}(\mu_\alpha^{(s)})} \Big|\frac{\partial}{\partial x}\big[v_\alpha^{(s)}(1-\varphi_\alpha^{(s)})\big]\Big|^m\,dx \leqslant$$

$$\leqslant C_{16}\;\mu_s \sum_{\alpha\in I'_{1,s}} |q_\alpha^{(s)}|^m\,[\lambda_s\rho_s]^n \leqslant C_{16}\;\mu_s \int\limits_\Omega |q_s(x)|^m\,dx\,,$$

and the right-hand side of the last inequality tends to zero as $s\to\infty$. Taking the assumption on $z_s(x)$ into account we obtain

$$\lim_{s\to\infty} J_s^{(1)} = 0. \tag{3.29}$$

The equality

$$J_s^{(2)} = 0 \tag{3.30}$$

follows from the definition of the functions $v_\alpha^{(s)}(x,q_\alpha^{(s)})$ (see (1.11) and (1.12)) and from the properties of $\eta_\alpha^{(s)}(x)$ and $z_s(x)$.

In order to estimate $J_s^{(3)}$ we remark that the inequality

$$|v_\alpha^{(s)}(x,q_\alpha^{(s)})| \leqslant \overline{\mu}_\alpha^{(s)},\quad \overline{\mu}_\alpha^{(s)} = C_{17}[\lambda_s\rho_s]^2\,|q_\alpha^{(s)}|^{m-1} \tag{3.31}$$

holds for $\alpha\in I'_{1,s}$ and $x\in\tilde K_s(\alpha)\setminus K\big(x_\alpha^{(s)},\frac{3\lambda_s\rho_s}{2}\big)$. We obtain this estimate using Lemma 2.3 and (3.11), taking into account that $|q_\alpha^{(s)}|^{m-1}\cdot\lambda_s\rho_s\leqslant 1$ for $\alpha\in I'_{1,s}$, which implies that the second condition in (2.4) is satisfied.

By condition A.2 and Hölder's inequality we obtain the estimate for $J_s^{(3)}$:

$$|J_s^{(3)}| \leqslant C_{18}\,\frac{1}{\lambda_s\rho_s}\,\Big\{\sum_{\alpha\in I'_{1,s}} \int\limits_{E_\alpha^{(s)}(\overline{\mu}_\alpha^{(s)})} \Big|\frac{\partial v_\alpha^{(s)}}{\partial x}\Big|^m\,dx\Big\}^{\frac{m-1}{m}} \cdot \Big\{[\lambda_s\rho_s]^m \int\limits_\Omega \Big|\frac{\partial z_s}{\partial x}\Big|^m\,dx +$$

$$+\int\limits_\Omega |z_s(x)|^m\,dx\Big\}^{\frac{1}{m}} + C_{18}\,\frac{1}{\lambda_s\rho_s}\Big\{\sum_{\alpha\in I'_{1,s}} \int\limits_{E_\alpha^{(s)}(\overline{\mu}_\alpha^{(s)})} \Big|\frac{\partial v_\alpha^{(s)}}{\partial x}\Big|^2\,dx\Big\}^{\frac{1}{2}}. \tag{3.32}$$

$$\cdot\{[\lambda_s\rho_s]^2\int\limits_\Omega |\frac{\partial z_s}{\partial x}|^2\,dx + \int\limits_\Omega |z_s(x)|^2\,dx\}^{\frac{1}{2}},$$

where $v_\alpha^{(s)} = v_\alpha^{(s)}(x, q_\alpha^{(s)})$ and $\overline{\mu}_\alpha^{(s)}$ is defined by (3.31). In the right-hand side of (3.22) the factors containing $z_s(x)$ can be estimated from above by $C_{19}\zeta_s$, where ζ_s is defined by (3.23).

In order to check the equality

$$\lim_{s\to\infty} J_s^{(3)} = 0, \tag{3.33}$$

it is sufficient to establish the estimate

$$J_s^{(4)} := \sum_{\alpha\in I'_{1,s}} \int\limits_{E_\alpha^{(s)}(\overline{\mu}_\alpha^{(s)})} \left(|\frac{\partial v_\alpha^{(s)}}{\partial x}|^2 + |\frac{\partial v_\alpha^{(s)}}{\partial x}|^m\right) dx \leqslant C_{20}[\lambda_s\rho_s]^2\cdot\zeta_s^{-\frac{m-2}{m-1}}. \tag{3.34}$$

This inequality follows from (2.2), (3.11), (3.14), (3.18), (3.31), and condition B:

$$\begin{aligned} J_s^{(4)} &\leqslant C_{21}\,[\lambda_s\rho_s]^2 \sum_{\alpha\in I'_{1,s}} |q_\alpha^{(s)}|^{2m-2}\,[\lambda_s\rho_s]^n \leqslant \\ &\leqslant C_{22}\,[\lambda_s\rho_s]^2\,\zeta_s^{-\frac{m-2}{m-1}}\cdot\int\limits_\Omega |q_s(x)|^m\,dx. \end{aligned}$$

This proves inequality (3.34) and concludes the proof of the Convergence Theorem.

4. Construction and properties of test functions

In this section we construct special functions which belong to the space $\overset{\circ}{W}{}^1_m(\Omega_s)$ and which will be used later as test functions in the integral identity corresponding to the boundary value problem (0.1), (0.2).

As in Section 3 we fix the sequences ρ_s, μ_s, λ_s introduced in (3.1), and the subdivision of the domain Ω defined by (1.13). For $s = 1, 2, ...$ and $\alpha\in I_s$ we define $I_s(\alpha)$ as the set of all multi-indices with integer coordinates such that $K(2\rho_s\beta, \rho_s)\subset K_s(\alpha)\setminus \overset{\circ}{K}{}'_s(\alpha)$, where $K_s(\alpha)$, $K'_s(\alpha)$ are the cubes defined in (3.2) and $\overset{\circ}{K}{}'_s(\alpha)$ is the interior of the cube $K'_s(\alpha)$. For $\beta\in I_s(\alpha)$ we set $x_{\alpha\beta}^{(s)} = 2\rho_s\beta$ and $K_s(\alpha,\beta) = K(x_{\alpha\beta}^{(s)}, \rho_s)$. Then we have the following decomposition:

$$K_s(\alpha)\setminus \overset{\circ}{K}{}'_s(\alpha) = \bigcup_{\beta\in I_s(\alpha)} K_s(\alpha,\beta). \tag{4.1}$$

Let $|I_s|$, $|I_s(\alpha)|$ be the numbers of multi-indices belonging to the sets I_s and $I_s(\alpha)$ respectively. It is easy to see that

$$|I_s| \leqslant C(\Omega)\,[\lambda_s \rho_s]^{-n}, \quad |I_s(\alpha)| \leqslant 2n\lambda_s^{n-1}, \tag{4.2}$$

where the constant $C(\Omega)$ depends only on the measure of Ω.

Let $g(x)$ be an arbitrary function of class $C_0^\infty(\Omega)$. Let us consider the sequence

$$q_s(x) = f_s(x) - u_0^{(s)}(x) - g(x), \tag{4.3}$$

where

$$f_s(x) = \frac{1}{[\lambda_s \rho_s]^n} \int\limits_{R^n} K\left(\frac{|x-y|}{\lambda_s \rho_s}\right) f(y)\,dy,$$
$$u_0^{(s)}(x) = \frac{1}{[\lambda_s \rho_s]^n} \int\limits_{R^n} K\left(\frac{|x-y|}{\lambda_s \rho_s}\right) u_0(y)\,dy,$$

$f(x)$ is the boundary function from (0.2), $u_0(x)$ is the weak limit of the sequence $u_s(x)$, solutions of the boundary value problem (0.1), (0.2), and the kernel $K(t)$ is the same as in (2.13).

We define new cut-off functions $\tilde{\varphi}_\alpha^{(s)}(x)$ by

$$\tilde{\varphi}_\alpha^{(s)}(x) = \frac{2}{\mu_\alpha^{(s)}} \min\left\{\left[v_\alpha^{(s)}(x,1) - \frac{\mu_\alpha^{(s)}}{2}\right]_+, \ \frac{\mu_\alpha^{(s)}}{2}\right\}, \tag{4.4}$$

where $v_\alpha^{(s)}(x,1)$ and $\mu_\alpha^{(s)}$ are the same as in (3.4) and (3.7). In accordance with [10], we can define two sequences of nonnegative functions $\chi_{\alpha\beta}^{(s)}(x)$, $\psi_{\alpha\beta}^{(s)}(x)$, for $\alpha \in I_s$, $\beta \in I_s(\alpha)$, such that the following properties are satisfied:

1) there exists a number s_2 such that the inclusions

$$\operatorname{supp}\, \chi_{\alpha\beta}^{(s)} \subset K\big(x_{\alpha\beta}^{(s)}, \frac{3\rho_s}{2}\big) \quad \text{for} \quad \alpha \in I_s, \ \beta \in I_s(\alpha) \tag{4.5}$$

holds for $s \geqslant s_2$ where supp $\chi_{\alpha\beta}^{(s)}$ is the support of the function $\chi_{\alpha\beta}^{(s)}(x)$;

2) for every point $x \in R^n$ in the sequence of numbers $\{\chi_{\alpha\beta}^{(s)}(x) : \alpha \in I_s, \ \beta \in I_s(\alpha)\}$ no more that 2^n numbers are non-zero and there exists a number K_5 depending only on m, n, ν_1, ν_2, A such that the inequalities

$$\chi_{\alpha\beta}^{(s)}(x) \leqslant K_5, \quad \int\limits_{R^n} \left|\frac{\partial \chi_{\alpha\beta}^{(s)}(x)}{\partial x}\right|^m dx \leqslant K_5\, \mu_s^{1-m} \cdot \rho_s^n \tag{4.6}$$

holds for $s = 1, 2, \ldots,\ \alpha \in I_s,\ \beta \in I_s(\alpha)$;

3) the functions $\psi^{(s)}_{\alpha\beta}(x)$ are defined by the equality

$$\psi^{(s)}_{\alpha\beta}(x) = \chi^{(s)}_{\alpha\beta}(x)\{1 - \sum_{\gamma\in I_s} \tilde{\varphi}^{(s)}_{\gamma}(x)\}, \quad x \in R^n; \tag{4.7}$$

4) the following equalities hold:

$$\sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \chi^{(s)}_{\alpha\beta}(x) = 1 \quad \text{for} \quad x \in \bigcup_{\alpha\in I_s} \bigcup_{\beta\in I_s(\alpha)} \{K_s(\alpha,\beta) \setminus \Omega_s\}\,, \tag{4.8}$$

$$\sum_{\alpha\in I_s} \tilde{\varphi}^{(s)}_{\alpha}(x) + \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \psi^{(s)}_{\alpha\beta}(x) = 1 \quad \text{for} \quad x \in \bigcup_{\alpha\in I_s} \{K_s(\alpha) \setminus \Omega_s\}\,. \tag{4.9}$$

We shall assume later that

$$s \geqslant \max\{s_1, s_2\}. \tag{4.10}$$

Remark that from inclusions (3.8) and (4.5) we obtain that for every $x \in R^n$, $\alpha, \gamma \in I_s$, $\beta \in I_s(\alpha)$ we have

$$\chi^{(s)}_{\alpha\beta}(x)\ \tilde{\varphi}^{(s)}_{\gamma}(x) = 0\,, \quad \chi^{(s)}_{\alpha\beta}(x)\ \varphi^{(s)}_{\gamma}(x) = 0 \quad \text{if} \quad \alpha \neq \gamma. \tag{4.11}$$

Let us introduce the sequence

$$h_s(x) = f(x) - q_s(x) + r_s(x) + \sum_{i=1}^{3} r^{(i)}_s(x), \tag{4.12}$$

where

$$\begin{aligned} r^{(1)}_s(x) &= \sum_{\alpha\in I_s} [q_s(x) - q^{(s)}_{\alpha}]\, \tilde{\varphi}^{(s)}_{\alpha}(x), \\ r^{(2)}_s(x) &= q_s(x) \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \psi^{(s)}_{\alpha\beta}(x), \\ r^{(3)}_s(x) &= \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} [q^{(s)}_{\alpha}\, \tilde{\varphi}^{(s)}_{\alpha}(x) - v^{(s)}_{\alpha}(x, q^{(s)}_{\alpha})\, \varphi^{(s)}_{\alpha}(x)]\, \chi^{(s)}_{\alpha\beta}(x), \end{aligned} \tag{4.13}$$

and the sequences $r_s(x)$ and $q_s(x)$ are defined by (1.14) and (4.3). The sequences $q^{(s)}_{\alpha}$ and $\varphi^{(s)}_s(x)$ are the same as in Section 3, with $q_s(x)$ defined by (4.3).

Lemma 4.1. *Assume that conditions A.1 and A.2 are satisfied, and let* $\overline{g}(x)$ *be an arbitrary function in the space* $C_0^\infty(\Omega)$*. Then there exists a number* $s_3(\overline{g})$*, depending on* $\overline{g}(x)$*, such that the inclusion*

$$\overline{g}(x)[u_s(x) - h_s(x)] \in \overset{\circ}{W}{}^1_m(\Omega_s) \tag{4.14}$$

holds for $s \geqslant \max\{s_1, s_2, s_3(\overline{g})\}$*.*

Proof. By the definition of the functions $v_\alpha^{(s)}(x, q_\alpha^{(s)})$, $\varphi_\alpha^{(s)}(x)$, Lemma 3.1, and inclusion (4.8) we obtain that the function

$$r_s^{(4)}(x) := \sum_{\alpha\in I_s} [v_\alpha^{(s)}(x, q_\alpha^{(s)})\, \varphi_\alpha^{(s)}(x) - q_\alpha^{(s)}\, \tilde{\varphi}_\alpha^{(s)}(x)] \cdot \{1 - \sum_{\gamma\in I_s} \sum_{\delta\in I_s(\gamma)} \chi_{\gamma\delta}^{(s)}(x)\} \tag{4.15}$$

belongs to $\overset{\circ}{W}{}^1_m(\Omega_s')$, where $\Omega_s' = \Omega \setminus \{ \bigcup\limits_{\alpha\in I_s} [K_s(\alpha) \setminus \Omega_s]\}$.

From (4.9) we obtain the inclusion

$$r_s^{(5)}(x) := q_s(x)\{1 - \sum_{\alpha\in I_s} \tilde{\varphi}_\alpha^{(s)}(x) - \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \psi_{\alpha\beta}^{(s)}(x)\} \in \overset{\circ}{W}{}^1_m(\Omega_s'). \tag{4.16}$$

Taking (4.11) into account we obtain

$$u_s(x) - h_s(x) = u_s(x) - f(x) - r_s^{(4)}(x) + r_s^{(5)}(x) \in \overset{\circ}{W}{}^1_m(\Omega_s').$$

Inclusion (4.14) follows now from the construction of the subdivision (1.13) of the domain Ω and from the choice of the function $\overline{g}(x)$. The proof of lemma is complete.

Lemma 4.2. *Assume that conditions A.1, A.2, and B are satisfied. Then the sequences* $r_s^{(i)}(x)$*,* $i = 1, 2, 3$*, defined by (4.13), converge strongly to zero in the space* $W_m^1(\Omega)$ *as* $s \to \infty$*.*

Proof. Assume that s is large enough so that inequalities (3.11) and (4.10) are satisfied. Using (2.2), (3.7), (4.4), and condition B we have the estimate

$$\left\| \frac{\partial \tilde{\varphi}_\alpha^{(s)}(x)}{\partial x} \right\|^m_{L_m(\Omega)} \leqslant 2^m\, [\mu_\alpha^{(s)}]^{-m} \int\limits_{\overline{E}_\alpha^{(s)}} \left| \frac{\partial v_\alpha^{(s)}(x, 1)}{\partial x} \right|^m dx \leqslant C_{23}\, \mu_s^{1-m}\, [\lambda_s \rho_s]^n, \tag{4.17}$$

where $\overline{E}_\alpha^{(s)} = \big\{x \in \Omega_0 : \mu_\alpha^{(s)}/2 \leqslant v_\alpha^{(s)}(x, 1) \leqslant \mu_\alpha^{(s)}\big\}$.

Let us estimate the norm of the gradient of $r_s^{(1)}(x)$ in $L_m(\Omega)$:

$$\begin{aligned} \left\| \frac{\partial}{\partial x} r_s^{(1)}(x) \right\|_{L_m(\Omega)}^m &\leqslant C_{24} \sum_{\alpha \in I_s} \int\limits_{G_\alpha^{(s)}} \left| \frac{\partial q_s(x)}{\partial x} \right|^m dx + \\ &+ C_{24} \sum_{\alpha \in I_s} \int\limits_{\Omega} |g(x) - g_\alpha^{(s)}|^m \cdot \left| \frac{\partial \tilde{\varphi}_\alpha^{(s)}(x)}{\partial x} \right|^m dx + \\ &+ C_{24} \sum_{\alpha \in I_s} \int\limits_{\Omega} \{ |f_s(x) - f_\alpha^{(s)}|^m + |u_0^{(s)}(x) - u_\alpha^{(s)}|^m \} \left| \frac{\partial \tilde{\varphi}_\alpha^{(s)}(x)}{\partial x} \right|^m dx, \end{aligned} \tag{4.18}$$

where $f_\alpha^{(s)}, u_\alpha^{(s)}, g_\alpha^{(s)}$ are the mean values of the functions $f_s(x), u_0^{(s)}(x), g(x)$ in the cube $K_s(\alpha)$. The first term in the right-hand side of (4.18) tends to zero as $s \to \infty$ by Lemma 3.2, the strong convergence of the sequence $q_s(x)$ in $W_m^1(\Omega)$, and the absolute continuity of the integral. Since the function $g(x)$ is smooth, the second term tends to zero by (4.17) and (3.1).

Using (4.17) and Lemma 2.4, the third term in the right-hand side of (4.18) can be estimated from above by

$$C_{25}\, \mu_s^{1-m} \, [\lambda_s \rho_s]^m \int\limits_{\Omega} \left[\left| \frac{\partial f(x)}{\partial x} \right|^m + \left| \frac{\partial u_0(x)}{\partial x} \right|^m \right] dx \, ,$$

which vanishes as $s \to \infty$ by (3.1). This completes the proof of the strong convergence of $r_s^{(1)}(x)$ to zero in $W_m^1(\Omega)$.

Let $\mathcal{D}_{\alpha\beta}^{(s)}$ be the support of the function $\psi_{\alpha\beta}^{(s)}(x)$. Then from (4.2), (4.5) and (4.7) we have

$$\sum_{\alpha \in I_s} \sum_{\beta \in I_s(\alpha)} \text{meas } \mathcal{D}_{\alpha\beta}^{(s)} \leqslant C_{26} \frac{1}{\lambda_s}. \tag{4.19}$$

We will use also the estimate

$$\int\limits_{\Omega} \left| \frac{\partial v_\alpha^{(s)}(x,q)}{\partial x} \right|^m \, [\chi_{\alpha\beta}^{(s)}(x)]^m \, dx \, \leqslant C_{27} \, [\mu_s^{1-m} |q|^m + 1] \, \rho_s^n, \tag{4.20}$$

which follows as in the proof of inequality (4.37) in [10]. From (4.7), (4.11) and from inequalities (4.6), (4.20) we obtain the estimate

$$\int\limits_{\Omega} \left| \frac{\partial \psi_{\alpha\beta}^{(s)}(x)}{\partial x} \right|^m dx \, \leqslant \, C_{28} \, \mu_s^{1-2m} \, \rho_s^n. \tag{4.21}$$

Let us estimate the norm of $r_s^{(2)}(x)$ in $W_m^1(\Omega)$. We rewrite $r_s^{(2)}(x)$ in the form

$$r_s^{(2)}(x) = \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \left\{[q_s(x) - \overline{q}_\alpha^{(s)}] + \overline{q}_\alpha^{(s)}\right\} \psi_{\alpha\beta}^{(s)}(x),$$

where $\overline{q}_\alpha^{(s)}$ is the mean value of the function $q_s(x)$ in the cube $\tilde{K}_s(\alpha) = K(x_\alpha^{(s)}, 2\lambda_s\rho_s)$. Using (4.6), (4.7), and (4.21) we obtain the inequality

$$\begin{aligned}\int_\Omega \left|\frac{\partial r_s^{(2)}(x)}{\partial x}\right|^m dx \leqslant C_{29} \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \int_{\mathcal{D}_{\alpha\beta}^{(s)}} \left|\frac{\partial q_s(x)}{\partial x}\right|^m dx + \\ + C_{29}\, \mu_s^{1-2m} \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \left|\overline{q}_\alpha^{(s)}\right|^m \cdot \rho_s^n + \\ + C_{29} \sum_{\alpha\in I_s} \int_{\tilde{K}_s(\alpha)} \left|q_s(x) - \overline{q}_\alpha^{(s)}\right|^m \sum_{\beta\in I_s(\alpha)} \left|\frac{\partial \psi_{\alpha\beta}^{(s)}(x)}{\partial x}\right|^m dx.\end{aligned} \tag{4.22}$$

The first term in the right-hand side of (4.22) tends to zero as $s \to \infty$ by (4.19) and the strong convergence of the sequence $q_s(x)$ in $W_m^1(\Omega)$. Using (4.2) and (3.14), the second term in the right-hand side of (4.22) is estimated from above by

$$C_{30}\, \lambda_s^{-1} \mu_s^{1-2m} \sum_{\alpha\in I_s} \left|\overline{q}_\alpha^{s)}\right|^m [\lambda_s\rho_s]^n \leqslant C_{31} \lambda_s^{-1} \mu_s^{1-2m} \int_\Omega |q_s(x)|^m\, dx,$$

which tends to zero by the choice of μ_s, λ_s.

Using Lemma 2.4 and inequalities (4.2), (4.21), the third term in the right-hand side of (4.22) is estimated from above by

$$\begin{aligned} C_{32}\, \lambda_s^{-1}\, \mu_s^{1-2m} [\lambda_s\rho_s]^m \Bigg\{ \int_\Omega \left[\left|\frac{\partial f(x)}{\partial x}\right|^m + \left|\frac{\partial u_0(x)}{\partial x}\right|^m\right] dx + \\ + \max_{x\in\Omega} \left|\frac{\partial g(x)}{\partial x}\right|^m \cdot \operatorname{meas}\Omega \Bigg\}, \end{aligned}$$

which converges to zero by (3.1). This concludes the proof of the strong convergence to zero of $r_s^{(2)}(x)$ in $W_m^1(\Omega)$.

The same property for $r_s^{(3)}(x)$ follows from the inequality

$$\begin{aligned}\int_\Omega \left|\frac{\partial r_s^{(3)}(x)}{\partial x}\right|^m dx \leqslant C_{33} \sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \left\{|q_\alpha^{(s)}|^m\, \mu_s^{1-2m} + 1\right\} \rho_s^n \leqslant \\ \leqslant C_{34} \lambda_s^{-1} \int_\Omega \left\{\mu_s^{1-2m}\, |q_s(x)|^m + 1\right\} dx\,,\end{aligned}$$

which can be obtained by using (4.2), (4.6), (4.20), and (3.14). This completes the proof of Lemma 4.2.

Let $g(x)$ be the same function as before, and let $g_\alpha^{(s)}$ be its mean value in the cube $K_s(\alpha)$. We introduce the sequence

$$g_s(x) = g(x) + \rho_s(x) + \sum_{i=1}^{3} \rho_s^{(i)}(x), \tag{4.23}$$

where

$$\begin{aligned}
\rho_s(x) &= -\sum_{\alpha\in I_s} \frac{1}{\tilde{q}_\alpha^{(s)}}\, v_\alpha^{(s)}(x, \tilde{q}_\alpha^{(s)})\, \varphi_\alpha^{(s)}(x)\, g_\alpha^{(s)},\\
\rho_s^{(1)}(x) &= \sum_{\alpha\in I_s} [g_\alpha^{(s)} - g(x)]\, \tilde{\varphi}_\alpha^{(s)}(x),\\
\rho_s^{(2)}(x) &= -g(x) \sum_{\alpha\in I_s} \sum_{\beta\in I_s} \psi_{\alpha\beta}^{(s)}(x),\\
\rho_s^{(3)}(x) &= -\sum_{\alpha\in I_s} \sum_{\beta\in I_s(\alpha)} \left[\tilde{\varphi}_\alpha^{(s)}(x) - \frac{1}{\tilde{q}_\alpha^{(s)}} v_\alpha^{(s)}(x, \tilde{q}_\alpha^{(s)})\, \varphi_\alpha^{(s)}(x)\right] g_\alpha^{(s)}\, \chi_{\alpha\beta}^{(s)}(x).
\end{aligned} \tag{4.24}$$

Here $\varphi_\alpha^{(s)}(x)$, $\tilde{\varphi}_\alpha^{(s)}(x)$, $\psi_{\alpha\beta}^{(s)}(x)$, $\chi_{\alpha\beta}^{(s)}(x)$ are the same functions as in (4.13),

$$\tilde{q}_\alpha^{(s)} = q_\alpha^{(s)} \quad \text{for} \quad \alpha \in I_s', \qquad \tilde{q}_\alpha^{(s)} = 2\mu_s \quad \text{for} \quad \alpha \in I_s'',$$

and $q_\alpha^{(s)}$ is the mean value in the cube $K_s(\alpha)$ of the function $q_s(x)$ defined by (4.3).

Lemma 4.3. *Assume that conditions of Lemma 4.1 are satisfied. Then there exists a number $s_4(\overline{g})$ depending on $\overline{g}(x)$ such that*

$$\overline{g}(x)\, g_s(x) \in \overset{\circ}{W}{}_m^1(\Omega_s) \tag{4.25}$$

for $s \geqslant s_4(\overline{g})$.

The proof is analogous with the proof of Lemma 4.1.

Lemma 4.4. *Assume that conditions A.1, A.2, and B are satisfied. Then the sequence $\rho_s(x)$ is bounded in $W_m^1(\Omega)$ and converges to zero strongly in $W_p^1(\Omega)$ for $p < m$.*

The proof is analogous with the proof of Lemma 3.3.

Lemma 4.5. *Assume that conditions A.1, A.2, and B are satisfied. Then the sequences $\rho_s^{(i)}(x)$, $i = 1, 2, 3$, converge strongly to zero in $W_m^1(\Omega)$ as $s \to \infty$.*

The proof is analogous with the proof of Lemma 4.2.

5. Construction of the limit boundary value problem

Using condition C we can conclude that for an arbitrary positive number ε there exist two positive numbers $r(\varepsilon)$ and $s(\varepsilon)$, and a sequence $r_s(\varepsilon)$ converging to zero as $s \to \infty$, such that the inequality

$$\left| \frac{1}{\operatorname{meas} K(x,r)} \, C_A(K(x,r) \setminus \Omega_s, q) - c(x,q) \right| < \varepsilon \tag{5.1}$$

holds for $s \geqslant s(\varepsilon), \quad r_s(\varepsilon) \leqslant r \leqslant r(\varepsilon), \quad |q| \leqslant \frac{1}{\varepsilon}, \quad x \in \Omega.$

Lemma 5.1. *Assume that conditions A.1, A.2, B, and C are satisfied. Then the function $c(x,q)$ defined by condition C satisfies the inequality*

$$|c(x,q)| \leqslant K_6 |q|^{m-1} \tag{5.2}$$

with a constant K_6 depending only on n, m, ν_1, ν_2, A.

Proof. Inequality (5.2) follows immediately from definition of the function $c(x,q)$ and inequality (2.2).

Proof of Theorem 1.2. Let us fix a positive number ε and let ρ_s be the sequence defined by

$$\rho_s = r_s + r_s(\varepsilon). \tag{5.3}$$

Let μ_s and λ_s be the sequences defined by equalities (3.1) with this particular choice of ρ_s.

We fix some function $g(x)$ in the space $C_0^\infty(\Omega)$ and choose a nonnegative function $\overline{g}(x) \in C_0^\infty(\Omega)$ such that

$$\overline{g}(x)\, g(x) = g(x), \qquad |g(x)| \leqslant 1, \qquad \overline{g}(x) \leqslant 1 \qquad \text{for} \quad x \in \Omega. \tag{5.4}$$

Using inequality (1.2) we get

$$0 \leqslant \sum_{j=1}^{n} \int\limits_{\Omega_s} \left[a_j\left(x, \frac{\partial u_s}{\partial x}\right) - a_j\left(x, \frac{\partial h_s}{\partial x}\right) \right] \frac{\partial (u_s - h_s)}{\partial x_j} \, \overline{g}(x)\, dx = \sum_{i=1}^{4} R_s^{(i)}\,, \tag{5.5}$$

where $u_s(x)$ is the solution of problem (0.1), (0.2), $h_s(x)$ in the function introduced in (4.12), and $R_s^{(i)}$ are defined by equalities

$$
\begin{aligned}
R_s^{(1)} &= -\sum_{j=1}^{n} \int\limits_{\Omega_s} \left[a_j\left(x, \frac{\partial u_s}{\partial x}\right) - a_j\left(x, \frac{\partial h_s}{\partial x}\right)\right] (u_s - h_s + g) \frac{\partial \overline{g}(x)}{\partial x_j}\, dx, \\
R_s^{(2)} &= \sum_{j=1}^{n} \int\limits_{\Omega_s} a_j\left(x, \frac{\partial u_s}{\partial x}\right) \frac{\partial}{\partial x_j} \{\overline{g}(x)[u_s(x) - h_s(x)]\}\, dx, \\
R_s^{(3)} &= -\sum_{j=1}^{n} \int\limits_{\Omega_s} \left[a_j\left(x, \frac{\partial h_s}{\partial x}\right) - a_j\left(x, \frac{\partial r_s}{\partial x}\right)\right] \frac{\partial}{\partial x_j} \{\overline{g}(x)[u_s(x) - h_s(x)]\}\, dx, \\
R_s^{(4)} &= -\sum_{j=1}^{n} \int\limits_{\Omega_s} a_j\left(x, \frac{\partial r_s}{\partial x}\right) \frac{\partial}{\partial x_j} \{\overline{g}(x)[u_s(x) - h_s(x)]\}\, dx.
\end{aligned}
\tag{5.6}
$$

By Lemmas 3.3 and 4.2 the sequence $u_s(x) - h_s(x) + g(x)$ converges to zero strongly in $L_m(\Omega)$ and then by the same Lemmas and (3.1) we obtain the equality

$$
\lim_{s\to\infty} R_s^{(1)} = 0. \tag{5.7}
$$

Using the definition of $u_s(x)$ and Lemmas 3.3, 4.1, 4.2 we get the equality

$$
\lim_{s\to\infty} R_s^{(2)} = -\sum_{j=1}^{n} \int\limits_{\Omega} f_j(x) \frac{\partial g(x)}{\partial x_j}\, dx. \tag{5.8}
$$

We check now that

$$
\lim_{s\to\infty} R_s^{(3)} = \sum_{j=1}^{n} \int\limits_{\Omega} a_j\left(x, \frac{\partial u_0}{\partial x} + \frac{\partial g}{\partial x}\right) \frac{\partial g}{\partial x}\, dx. \tag{5.9}
$$

By inequality (1.3) and Lemmas 3.3, 4.2 we obtain that the sequence

$$
b_j^{(s)}(x) = a_j\left(x, \frac{\partial h_s(x)}{\partial x}\right) - a_j\left(x, \frac{\partial r_s(x)}{\partial x}\right)
$$

converges in measure to the function $a_j\left(x, \frac{\partial u_0}{\partial x} + \frac{\partial g}{\partial x}\right)$. To prove that this sequence converges strongly in $L_{\frac{m}{m-1}}(\Omega)$, it is sufficient to establish that the integrals

$$
\int\limits_{\Omega} \left| b_j^{(s)}(x)\right|^{\frac{m}{m-1}} dx, \quad s = 1, 2, \ldots \tag{5.10}
$$

satisfy the absolute continuity property uniformly with respect to s.

Using (1.3) and Hölder's inequality we obtain the estimate

$$\int\limits_E \left|b_j^{(s)}(x)\right|^{\frac{m}{m-1}} dx \leqslant C_{35} \left\{ \int\limits_E \left[1 + \left|\frac{\partial h_s(x)}{\partial x}\right| + \left|\frac{\partial r_s(x)}{\partial x}\right|\right]^m dx \right\}^{\frac{m-2}{m-1}} \cdot$$
$$\cdot \left\{ \int\limits_E \left|\frac{\partial h_s(x)}{\partial x} - \frac{\partial r_s(x)}{\partial x}\right|^m dx \right\}^{\frac{1}{m-1}}$$

for an arbitrary subset E of the set Ω. The last inequality and Lemma 4.2 guarantee the uniform absolute continuity for the sequence of integrals (5.10), and hence the strong convergence of $b_j^{(s)}(x)$ in $L_{\frac{m}{m-1}}(\Omega)$. Using this property and Lemmas 3.3, 4.2 we obtain equality (5.9).

We transform the term $R_s^{(4)}$ in the following way:

$$R_s^{(4)} = \sum_{i=5}^{9} R_s^{(i)}\,, \tag{5.11}$$

where

$$R_s^{(5)} = -\sum_{j=1}^n \int\limits_\Omega a_j\left(x, \frac{\partial r_s}{\partial x}\right) \frac{\partial}{\partial x_j} \left\{\overline{g}(x)[u_s(x) - h_s(x) + g_s(x)]\right\} dx,$$
$$R_s^{(6)} = \sum_{j=1}^n \int\limits_\Omega a_j\left(x, \frac{\partial r_s}{\partial x}\right) \frac{\partial g(x)}{\partial x_j}\, dx,$$
$$R_s^{(7)} = \sum_{j=1}^n \int\limits_\Omega a_j\left(x, \frac{\partial r_s}{\partial x}\right) \frac{\partial \overline{g}(x)}{\partial x_j} \cdot \rho_s(x)\, dx,$$
$$R_s^{(8)} = \sum_{j=1}^n \int\limits_\Omega a_j\left(x, \frac{\partial r_s}{\partial x}\right) \frac{\partial \rho_s(x)}{\partial x_j} \cdot \overline{g}(x)\, dx,$$
$$R_s^{(9)} = \sum_{i=1}^3 \sum_{j=1}^n \int\limits_\Omega a_j\left(x, \frac{\partial r_s}{\partial x}\right) \frac{\partial}{\partial x_j} \left[\overline{g}(x)\rho_s^{(i)}(x)\right] dx,$$

and the functions $g_s(x), \rho_s(x), \rho_s^{(i)}(x)$ are defined by (4.23), (4.24).

By virtue of Lemmas 3.3, 4.1–4.5 the sequence $z_s(x) = \overline{g}(x)[u_s(x) - h_s(x) + g_s(x)]$ satisfies the following conditions for s large enough: $z_s(x) \in \overset{\circ}{W}{}^1_m(\Omega_s)$ and $z_s(x)$ converges to zero weakly in $W_m^1(\Omega)$. Then by the Convergence Theorem 1.1 we obtain

$$\lim_{s\to\infty} R_s^{(5)} = 0. \tag{5.12}$$

Using (1.3) and Lemmas 3.3, 4.4. 4.5 we have

$$\lim_{s\to\infty} R_s^{(6)} = \lim_{s\to\infty} R_s^{(7)} = \lim_{s\to\infty} R_s^{(9)} = 0. \tag{5.13}$$

It remains to study the behaviour of $R_s^{(8)}$. From the definitions of $r_s(x), \rho_s(x)$ we obtain for $s \geqslant s_1$:

$$R_s^{(8)} = -\sum_{i=10}^{13} R_s^{(i)}\,, \tag{5.14}$$

where

$$R_s^{(10)} = \sum_{\alpha\in I_s}\sum_{j=1}^{n} \frac{g_\alpha^{(s)}}{\tilde{q}_\alpha^{(s)}} \int\limits_{\Omega_0} a_j\left(x, \frac{\partial\left(v_\alpha^{(s)}\varphi_\alpha^{(s)}\right)}{\partial x}\right)\cdot\frac{\partial}{\partial x_j}\left(\tilde{v}_\alpha^{(s)}\varphi_\alpha^{(s)}\right)\cdot\left[\overline{g}(x)-\overline{g}_\alpha^{(s)}\right]\,dx,$$

$$R_s^{(11)} = \sum_{\alpha\in I_s}\sum_{j=1}^{n} \frac{g_\alpha^{(s)}\overline{g}_\alpha^{(s)}}{\tilde{q}_\alpha^{(s)}} \int\limits_{\Omega_0} \left\{a_j\left(x, \frac{\partial\left(v_\alpha^{(s)}\varphi_\alpha^{(s)}\right)}{\partial x}\right)\cdot\frac{\partial}{\partial x_j}\left(\tilde{v}_\alpha^{(s)}\varphi_\alpha^{(s)}\right) - \right.$$
$$\left. - a_j\left(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\right)\frac{\partial \tilde{v}_\alpha^{(s)}}{\partial x_j}\right\}dx,$$

$$R_s^{(12)} = \sum_{\alpha\in I_s'}\sum_{j=1}^{n} \frac{g_\alpha^{(s)}\overline{g}_\alpha^{(s)}}{q_\alpha^{(s)}} \int\limits_{\Omega_0} a_j\left(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\right)\frac{\partial v_\alpha^{(s)}}{\partial x_j}\,dx,$$

$$R_s^{(13)} = \sum_{\alpha\in I_s''}\sum_{j=1}^{n} \frac{g_\alpha^{(s)}\overline{g}_\alpha^{(s)}}{2\mu_s} \int\limits_{\Omega_0} a_j\left(x, \frac{\partial v_\alpha^{(s)}}{\partial x}\right)\frac{\partial \tilde{v}_\alpha^{(s)}}{\partial x_j}\,dx;$$

here $v_\alpha^{(s)} = v_\alpha^{(s)}(x, q_\alpha^{(s)})$, $\tilde{v}_\alpha^{(s)} = v_\alpha^{(s)}(x, \tilde{q}_\alpha^{(s)})$, $\overline{g}_\alpha^{(s)}$ is the mean value of the function $\overline{g}(x)$ in the cube $K_s(\alpha)$, and the sets I_s', I_s'' are defined by (3.3).

Using inequalities (1.3), (2.2), (3.14), (3.15), condition B, and the smoothness of the function $\overline{g}(x)$, we obtain

$$\lim_{s\to\infty} R_s^{(10)} = 0. \tag{5.15}$$

We check now that

$$\lim_{s\to\infty} R_s^{(11)} = 0, \quad \lim_{s\to\infty} R_s^{(13)} = 0. \tag{5.16}$$

The first equality in (5.16) is established as in the proof of equality (3.29). We only need to observe that, by (2.2) and condition B, we have the estimate

$$\sum_{\alpha\in I_s} \frac{1}{|\tilde{q}_\alpha^{(s)}|^m} \int\limits_{\Omega_0} \left\{\left|\frac{\partial v_\alpha^{(s)}(x, q_\alpha^{(s)})}{\partial x}\right|^m + \left|\frac{\partial v_\alpha^{(s)}(x, \tilde{q}_\alpha^{(s)})}{\partial x}\right|^m\right\}dx \leqslant C_{36}.$$

The second equality in (5.16) follows immediately from estimate (2.2) and from the definition of I''_s.

In the rest of the paper we use the notation $\delta_i(t)$, $\gamma_i(t)$, $i = 1, 2, \dots$, to indicate nonnegative functions on R^1 satisfying the conditions

$$\lim_{t\to 0} \delta_i(t) = 0, \quad \lim_{t\to\infty} \gamma_i(t) = 0. \tag{5.17}$$

Lemma 5.2. *Assume that conditions A.1, A.2, B, and C are satisfied. Then there exist functions $\delta_1(t)$, $\gamma_1(t)$ satisfying conditions (5.17) such that*

$$\begin{aligned} &\Big| \sum_{\alpha\in I'_s} \sum_{j=1}^{n} \frac{\overline{g}_\alpha^{(s)} g_\alpha^{(s)}}{q_\alpha^{(s)}} \int\limits_{\Omega_0} a_j \left(x, \frac{\partial v_\alpha^{(s)}}{\partial x} \right) \frac{\partial v_\alpha^{(s)}}{\partial x_j}\, dx - \\ &- \int\limits_{\Omega} c(x, q_0(x))\, g(x)\, dx \Big| \leqslant \gamma_1(s) + \delta_1(\varepsilon)\,, \end{aligned} \tag{5.18}$$

where ε is the number fixed in the definition of ρ_s in (5.3), and $q_0(x) = f(x) - u_0(x) - g(x)$.

Proof. Define the sets of multi-indices

$$I_s(\varepsilon) = \left\{ \alpha \in I'_s : |q_\alpha^{(s)}| \leqslant \frac{1}{\varepsilon} \right\}, \quad J_s(\varepsilon) = \left\{ \alpha \in I'_s : |q_\alpha^{(s)}| > \frac{1}{\varepsilon} \right\},$$

and let $Q_s(\varepsilon)$ be the union of all cubes $K_s(\alpha)$ with $\alpha \in J_s(\varepsilon)$. As in (3.16) and (3.21) we obtain the estimates

$$\operatorname{meas} Q_s(\varepsilon) \leqslant C_{37}\, \varepsilon^m \int\limits_{\Omega} |q_s(x)|^m\, dx, \tag{5.19}$$

$$\begin{aligned} &\left| \sum_{\alpha\in J_s(\varepsilon)} \sum_{j=1}^{n} \frac{\overline{g}_\alpha^{(s)} g_\alpha^{(s)}}{q_\alpha^{(s)}} \int\limits_{\Omega_0} a_j \left(x, \frac{\partial v_\alpha^{(s)}}{\partial x} \right) \frac{\partial v_\alpha^{(s)}}{\partial x_j}\, dx \right| \leqslant \\ &\leqslant C_{38} \int\limits_{Q_s(\varepsilon)} |q_0(x)|^{m-1}\, dx + C_{38} \int\limits_{\Omega} |q_s(x) - q_0(x)|^{m-1}\, dx \leqslant \delta_2(\varepsilon) + \gamma_2(s). \end{aligned} \tag{5.20}$$

Using notation (1.16) and inequality (5.1) we have the estimate

$$\begin{aligned} &\Big| \sum_{\alpha\in I_s(\varepsilon)} \sum_{j=1}^{n} \frac{\overline{g}_\alpha^{(s)} g_\alpha^{(s)}}{q_\alpha^{(s)}} \int\limits_{\Omega_0} a_j \left(x, \frac{\partial v_\alpha^{(s)}}{\partial x} \right) \frac{\partial v_\alpha^{(s)}}{\partial x_j}\, dx - \\ &- \sum_{\alpha\in I_s(\varepsilon)} \overline{g}_\alpha^{(s)} g_\alpha^{(s)} c(x_\alpha^{(s)}, q_\alpha^{(s)}) \operatorname{meas} K'_s(\alpha) \Big| < \varepsilon \operatorname{meas} \Omega\,, \end{aligned} \tag{5.21}$$

provided s is so large that $\lambda_s \rho_s < r(\varepsilon)$.

In view of the continuity of the functions $c(x,q), g(x), \overline{g}(x)$ we obtain the inequality

$$\left| \sum_{\alpha \in I_s(\varepsilon)} \overline{g}_\alpha^{(s)} g_\alpha^{(s)} c(x_\alpha^{(s)}, q_\alpha^{(s)}) \operatorname{meas} K'_s(\alpha) - \sum_{\alpha \in I_s(\varepsilon)} \int\limits_{K'_s(\alpha)} c(x, q_\alpha^{(s)}) g(x) dx \right| \leqslant \gamma_3(s). \tag{5.22}$$

Using inequalities (5.2), (3.14), and (5.19) we have the following estimates

$$\left| \sum_{\alpha \in J_s(\varepsilon)} \int\limits_{K'_s(\alpha)} c(x, q_\alpha^{(s)}) g(x) dx \right| \leqslant C_{39} \varepsilon \int\limits_{\Omega} |q_s(x)|^m \, dx \leqslant \delta_3(\varepsilon), \tag{5.23}$$

$$\left| \sum_{\alpha \in I''_s} \int\limits_{K'_s(\alpha)} c(x, q_\alpha^{(s)}) g(x) dx \right| \leqslant C_{40} \, \mu_s^{m-1} \operatorname{meas} \Omega \leqslant \gamma_4(s). \tag{5.24}$$

From the last two estimates we obtain the inequality

$$\left| \sum_{\alpha \in I_s(\varepsilon)} \int\limits_{K'_s(\alpha)} c(x, q_\alpha^{(s)}) \, g(x) \, dx - \int\limits_{\Omega} c\left(x, q'_s(x)\right) \, g(x) \, dx \right| \leqslant \delta_3(\varepsilon) + \gamma_4(s) \,, \tag{5.25}$$

where

$$q'_s(x) = \sum_{\alpha \in I_s} q_\alpha^{(s)} \, \chi(K'_s(\alpha)) \,, \tag{5.26}$$

and $\chi(K'_s(\alpha))$ is the characteristic function of the set $K'_s(\alpha)$.

We check that the sequence $q'_s(x)$ defined by (5.26) converges to $q_0(x)$ strongly in $L_m(\Omega)$. Using Poincaré's inequality and (4.2) we have

$$\begin{aligned} \int\limits_{\Omega} |q'_s(x) - q_0(x)|^m \, dx \leqslant C_{41} \Bigg\{ & \int\limits_{U_s} |q_0(x)|^m \, dx + \\ & + \int\limits_{\Omega} |q_s(x) - q_0(x)|^m \, dx + \sum_{\alpha \in I_s} \int\limits_{K_s(\alpha) \setminus K'_s(\alpha)} |q_0(x)|^m \, dx + \\ & + (\lambda_s \rho_s)^m \int\limits_{\Omega} \left| \frac{\partial q_s(x)}{\partial x} \right|^m dx \Bigg\} \leqslant \gamma_5(s). \end{aligned} \tag{5.27}$$

From (5.2), (5.27) and the continuity of the function $c(x,q)$ we obtain the estimate

$$\left| \int\limits_{\Omega} [c(x, q'_s(x)) - c(x, q_0(x)] \; g(x) \, dx \right| \leqslant \gamma_6(s). \tag{5.28}$$

Now inequality (5.18) follows from (5.20)–(5.22), (5.25), (5.28) and the proof of the lemma is complete.

Inequality (5.5), together with (5.7)–(5.9), (5.11)–(5.16), and (5.18), implies

$$\begin{aligned}\sum_{j=1}^{n}\int_{\Omega}\left[a_j\left(x,\frac{\partial u_0}{\partial x}+\frac{\partial g}{\partial x}\right)-f_j(x)\right]\frac{\partial g(x)}{\partial x_j}\,dx - \\ -\int_{\Omega} c(x,f(x)-u_0(x)-g(x))g(x)dx \leqslant \gamma_7(s)+\delta_4(\varepsilon).\end{aligned} \tag{5.29}$$

In (5.29) the left-hand side is independent of s and ε while the right-hand side can be made arbitrarily small for sufficiently large s and sufficiently small ε. Hence, we obtain the inequality

$$\begin{aligned}\sum_{j=1}^{n}\int_{\Omega}\left[a_j\left(x,\frac{\partial u_0}{\partial x}+\frac{\partial g}{\partial x}\right)-f_j(x)\right]\frac{\partial g(x)}{\partial x_j}\,dx - \\ -\int_{\Omega} c(x,f(x)-u_0(x)-g(x))g(x)dx \leqslant 0\end{aligned} \tag{5.30}$$

for an arbitrary function $g(x)\in C_0^\infty(\Omega)$.

In (5.30) we can replace $g(x)$ by $\lambda g(x)$, with $\lambda>0$. Dividing both terms of (5.30) by λ and passing to the limit as $\lambda\to 0$ we obtain

$$\sum_{j=1}^{n}\int_{\Omega}\left[a_j\left(x,\frac{\partial u_0}{\partial x}\right)-f_j(x)\right]\frac{\partial g(x)}{\partial x_j}\,dx-\int_{\Omega} c(x,f(x)-u_0(x))g(x)dx \geqslant 0. \tag{5.31}$$

This inequality is true for both functions $g(x)$ and $-g(x)$ and consequently the left-hand side of (5.31) is equal zero for $g(x)\in C_0^\infty(\Omega)$. By an approximation argument we obtain the equality

$$\sum_{j=1}^{n}\int_{\Omega}\left[a_j\left(x,\frac{\partial u_0}{\partial x}\right)-f_j(x)\right]\frac{\partial g(x)}{\partial x_j}\,dx-\int_{\Omega} c(x,f(x)-u_0(x))g(x)dx = 0$$

for an arbitrary function $g(x)\in \overset{\circ}{W}{}^1_m(\Omega)$. Thus we have established that $u_0(x)$ is a solution of equation (1.18). The inclusion $u_0(x)\in f(x)+\overset{\circ}{W}{}^1_m(\Omega)$ follows immediately from $u_s(x)\in f(x)+\overset{\circ}{W}{}^1_m(\Omega)$. This completes the proof of Theorem 1.2.

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